504: Stochastic Processes.

Prerequisites: 471 and 403 (Real analysis and probability)

Reading:

1) Le Gall: Measure Theory, Probability and Stochastic Processes.
2) Khoshnevisan, Probability, AMS
3) Durrett, Probability: Theory and Examples, version 5a. Downloadable.

Topics:

1) Martingales
2) Discrete-time Markov Chains
3) Continuous-time processes
4) Brownian Motion
5) Markov Chains + Mixing Times

Remarks: We used to use Khoshnevisan but it has lots of typos and Le Gall more or less seems to have built on Khoshnevisan's work.

Khoshnevisan's Ch 1 and 2 are too nonrigorous to be useful, in my experience. But it has an amazing collection of problems and an amazing set of historical references.

Adminstrative Stuff

50 %	Midterm
50 %	Final

Generated σ-algebras (6.13)

$X_i : \Omega \longrightarrow \mathbb{R}^d$, $\{X_i\}_{i \in I}$ is the smallest σ-algebra wrt to which X_i are measurable.
we write it as $\sigma(\{X_i\}_{i \in I})$

Independence :

1) Events : $E_1, \dots E_n$ are indep
if

2) σ algebras : A collection of σ-algebras $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ are indep if any finite collection of events, one from each $\mathcal{F}_\alpha$ are independent.

3) Random variables $\{X_\alpha\}_{\alpha \in I}$ are independent if $\{\sigma(X_\alpha)\}_{\alpha \in I}$ are indep.

Tail σ-algebras let $\{X_i\}_{i=1}^{\infty}$ be a collection of rvs. $\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(\{X_i\}_{i=n}^{\infty})$

Kolmogorov's 0-1 law

If $\{X_i\}_{i=1}^{\infty}$ are iid, then their tail σ-algebra $\mathcal{T}$ is trivial: $\forall E \in \mathcal{T}$, $\mathbb{P}(E) = 0$ or 1.

$\Rightarrow$ Any $\mathcal{T}$ meas. function is a constant a.s.

Borel-Cantelli Lemma

Let $\{A_i\}_{i=1}^{\infty}$ be a collection of events. If

$$\sum_{n=1}^{\infty} P(A_n) < \infty$$

Interpretation?

then $P(A_n \text{ i.o.}) = 0$ i.o = infinitely often

$\Leftrightarrow \sum_{n=1}^{\infty} 1_{A_n} < \infty$ a.s

Converse: if $\{A_i\}$ are pairwise independent,

then $\sum_{n=1}^{\infty} P(A_n) = \infty \Rightarrow \sum_{n=1}^{\infty} 1_{A_n} = \infty$ a.s.

Radon-Nikodym Theorem

Def (Abs continuity) Given μ & ν on signed measures. $(\Omega, \mathcal{F})$ we say ν is abs continuous wrt μ $\nu \ll \mu$ if $\nu(A) = 0$ $\forall A \in \mathcal{F}$ st $\mu(A) = 0$

Ex: If $f \in L^1(\mu)$ $\nu(A) = \int_A f \, d\mu$ defines such a meas.

Radon-Nikodym Theorem: if $\nu \ll \mu$ are two finite (signed) measures on $(\Omega, \mathcal{F})$ then $\exists$ $\Pi_* \in L^1(\mu)$ st

$$\int f \, d\nu = \int f \, \Pi_* \, d\mu \quad \text{for all bounded meas. fns}$$

$f : \Omega \longrightarrow \mathbb{R}$

Ex: let $\Sigma \subset \mathcal{F}$ and let $\nu = \mu|_\Sigma$ (restriction)

Then clearly $\nu \ll \mu$. Further if $\mu(\Omega) = 1$, then $\nu(\Omega) = 1$

Thus they are both probability measures. $\Rightarrow \exists$ some Π_*

$$\int f \, d\nu = \int f \, \Pi_* \, d\mu$$

One proof is "merely" projection in L^2 and then approximation.

Kolmogorov Extension Theorem

To talk about stochastic processes $\{X_n\}_{n=0}^{\infty}$ on ∞ time intervals, we need to build measures on $\mathbb{R}^{\infty}$ or S^{∞} where S is the state space for the process X.

Borel σ-algebra: $\mathcal{B}(\mathbb{R}^{\infty})$. $\mathcal{B}(\mathbb{R}^n)$ is generated by open sets in $\mathbb{R}^n$. To generate $\mathcal{B}(\mathbb{R}^{\infty})$ we need

Cylinder sets: A cylinder set is of the form $A = A_1 \times A_2 \times \cdots$ where only finitely many A_i are non-empty.

A cylinder set is open if all the sets A_i are either open or empty.
$\mathcal{B}(\mathbb{R}^{\infty})$ is the smallest σ-algebra that contains all the open cylinder sets.

A family $\{(\mathbb{R}^n, \mathcal{B}^n, P^n)\}_{n=1}^{\infty}$ of probability measures is consistent if

$$P^n(A_1 \times \cdots \times A_n) = P^{n+1}(A_1 \times \cdots \times A_n \times \mathbb{R}^n) \quad \forall n \geq 1$$

and for all $A_i \in \mathcal{B}(\mathbb{R})$

If A is n dimensional, you're saying

$$P^{n+1}(\Pi_{n+1}^{-1}(A)) = P^n(A)$$

Kolmogorov Extension Theorem

Suppose $\{P^n\}_{n=0}^{\infty}$ is a consistent family of prob. measures on each $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then $\exists!$ a probability P^∞ on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ such that P^∞ projects correctly:

$$P^\infty(\pi_n^{-1}(B)) = P^n(B) \quad \forall B \in \mathcal{B}(\mathbb{R}^n)$$

Monotone Class Theorem

Set E, 2^E is the power set. If $\mathcal{C} \subset 2^E$, then $\sigma(\mathcal{C})$ is the intersection of all σ-algebras containing $\mathcal{C}$

<u>Definition</u>: A collection of sets $M \subset 2^E$ is called a monotone class if it is closed under increasing unions and relative complements.

Ring of sets: Collections of sets closed under relative complements and unions.
($\Rightarrow$ closure under intersection)
These ideas were needed in the Caratheodory extension Theorem.

Def: 1) $E \in M$
2) If $A, B \in M$, $A \subset B$, $B \setminus A \in M$
3) Let $A_n \uparrow$, then $\bigcup_n A_n \in M$

If $\mathcal{C} \subset 2^E$ is a collection, then $M(\mathcal{C})$ is the smallest monotone class containing $\mathcal{C}$.

<u>Lemma</u> If $\mathcal{C} \subset 2^E$ is stable under finite intersections then $M(\mathcal{C}) = \sigma(\mathcal{C})$

Π-λ Theorem (Durrett online version 5a, page 414, Theorem A.1.4)

The Π-λ theorem is due to Dynkin. This is simply another name for the Monotone class lemma.

Π-system: P is a Π-system if it is closed under finite intersection

λ-system: Same as monotone class of sets.

Theorem: If P is a Π-system and $\mathcal{L}$ is a λ-system containing P, then $\sigma(P) \subset \mathcal{L}$

☆ Show that this is equivalent to the monotone class lemma. Hint: consider $M(P)$ the smallest λ-system containing P

☆ Suppose $X_1, \dots, X_n$ are rvs such that

$$\mathbb{P}(X_1 \le x_1, \dots, X_n \le x_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \le x_i)$$

Show that $X_1, \dots, X_n$ are independent.

Durrett also uses the name Monotone class theorem to mean a version of the Monotone class lemma.

I don't really know what the history is, it doesn't matter too much.